Problem 7B,6
Suppose (X, S, μ) is a measure space and $0 < p < 1$. Prove that

$$\|f+g\|_p^p \leq \|f\|_p^p + \|f\|_p^p$$

for every S-measurable function $f, g : X \to F$.

Proof. Note that $(a+b)^p \leq a^p + b^p$ and then the result follows. □

Problem 7B,11
Suppose $1 \leq p \leq \infty$. Prove that

$$\{(a_1, a_2, \dots) : a_k \neq \ 0 \ for \ every \ k \in Z^+\}$$

is not an open set of l^p.

Proof. We only prove the case of $p = 2$. Other cases are the same. Take $a_k = \frac{1}{k}$. Denote it by a. We need to show for any $\epsilon > 0$, the ball $B(a, \epsilon)$ contains elements which are not in the above set. Take $N > \frac{1}{\epsilon}$. Choose b such that for $k \neq N, b_k = a_k; k = N, b_k = 0$. Then $\|a - b\|_p \leq \frac{1}{N} \leq \epsilon$. □

Problem 7B,15
Let

$$c_0 = \{(a_1, a_2, \dots) : \lim_{k\to\infty} a_k = 0\}.$$

Give c_0 the norm inherited from l^∞. Prove that

- c_0 is a Banach space.
- Prove that the dual space of c_0 can be identified as l^1.

Proof.
- It is easy to check that c_0 is closed subset of l^∞ so it is a Banach space.
- First given an element $b = (b_1, b_2, \dots)$, it acts on c_0 by: for any $a = (a_1, a_2, \dots) \in c_0, b(a) = \sum_{k=1}^\infty a_k b_k$. It is then easy to check that this is a lnear bounded functional on c_0 which the norm is bounded by $\|b\|_1$. Conversely, given a bounded linear functional f on c_0. For any positive integer k, set $e_k \in c_0$ whose only non-zero entry is the k-th entry and the value is 1. Then $\|e_k\|_\infty = 1$. Let $b_k = f(e_k)$. Note for any positive integer m,

$$\sum_{k=1}^m |b_k| = f(\sum_{k=1}^m sign(f(e_k))e_k) \leq \|f\|$$

where we have used that $\|\sum_{k=1}^m sign(f(e_k))e_k\|_\infty = 1$. Thus $b = (b_1, b_2, \dots) \in l^1$ and we can check by linearity and limit argument that $f(a) = \sum_{k=1}^\infty a_k b_k$.
□

Problem 8A,1
Let V be the lnear space of bounded continuous function from $\mathbb{R}$ to $\mathbb{F}$. Let $r_1, r_2, \dots$ be a list of rational numbers. Define

$$< f, g >= \sum_{k=1}^\infty \frac{f(r_k)g(\bar{r}_k)}{2^k}.$$

Show that $< .,. >$ is an inner product on V.

Proof. By the boundedness of f, g, the series is well defined. Then it is to check by definition that this defines an inner product. □

Problem 8A,5
Prove that

$$16 \leq (a+b+c+d)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d})$$

for any positive a, b, c, d with equality holds when $a = b = c = d$.

Proof. Only need to note the following general identity:

$$(\sum_{k=1}^{m} x_i y_i)^2 = (\sum_{k=1}^{m} x_i^2)(\sum_{k=1}^{m} y_i^2) - (\sum_{1 \leq i < j \leq m} (x_i y_j - x_j y_i)^2)$$

□

Problem 8A,7
Suppose f, g are elements of an inner product space and $\|f\| \leq 1, \|g\| \leq 1$. Prove that

$$\sqrt{1-\|f\|^2}\sqrt{1-\|g\|^2} \leq 1 - | < f, g > |$$

Proof. Note that by Cauchy-Schwarz inequality, the right hand side is greater than $1-\|f\|\|g\|$. So we only need to show

$$\sqrt{1-\|f\|^2}\sqrt{1-\|g\|^2} \leq 1-\|f\|\|g\|$$

Take square of both side and rearrange the term, this is equivalent to

$$2\|f\|\|g\| \leq \|f\|^2\|g\|^2.$$

This completes the proof. □

Problem 8A,15
Suppose f, g, h are elements of an inner product space. Prove that

$$\|h - \frac{1}{2}(f+g)\|^2 = \frac{\|h-f\|^2+\|h-g\|^2}{2} - \frac{\|f-g\|^2}{4}$$

Proof. Direct computation by 8.20. □

Problem 8B,11
Suppose V is a Hilbert space. A closed half-sapce is a set of the form

$$\{g : Re < g, h > \geq c\}$$

for some $h \in V, c \in \mathbb{R}$. Prove that every closed convex subset of V is the intersection of all the closed half-space containing it.

Proof. Let K be an closed convex subset of V and $\bar{K}$ be the intersection of all the closed half-space containing it. Obviouly $K \subset \bar{K}$ and $\bar{K}$ is also a closed convex subset. If there exists $p \in \bar{K} - K$. Then we can find an element $p_1 \in K$ such that $\|p - p_1\| = dist(p, K)$. Then for any $h \in K$, define the function $f(t) = \|th + (1-t)p_1 - p\|$ is a function defined on $t \in [0, 1]$ sucht that f attain its minimum at $t = 0$. Thus the right derivative of f at 0 is positive. This implies for any $h \in K$

$$Re < h, p_1 - p > \geq Re < p_1, p_1 - p >$$

Thus K lies in the half-space $\{g : Re < g, p_1 - p >\geq Re < p_1, p_1 - p >\}$. Note that

$$Re < p - p_1, p_1 - p >< 0.$$

which implies p does not lie in the half-space $\{g : Re < g, p_1 - p >\geq Re < p_1, p_1 - p >\}$. This contradiction finishes the proof. □

Problem 8B,13
In the real Bnanach space $\mathbb{R}^2$ with norm defined by $\|(x, y)\|_\infty = \max\{|x|, |y|\}$. Give an example of closed convex subset $U \in \mathbb{R}^2$ and $z \in \mathbb{R}$ such that there exists infinite choice of $w \in U$ with $\|z - w\| = dist(z, U)$.

Proof. Just take $U = [-1, 1]^2$ and $z = (2, 0)$. Then we can check w can be any $(1, t), t \in [-1, 1]$. □

Problem 8B,22
Prove that if v is a Hilbert space and $T : V \to V$ is a bounded linear map such that the dimension of range T is 1. Then there exist $g, h \in V$ such that for any $f \in V$

$$Tf =< f, g > h$$

Proof. It follows easily that kernel of T is of codimensional 1. Take V_1 to be the kernel. Then we can find its orthogonal complement spanned by an unit element $g \in V$. Then for any $f \in V$, $f = f_1+ < f, g > g$ where $f_1 \in V_1$. So

$$Tf = T(f_1+ < f, g > g) =< f, g > Tg$$

and this completes the proof. □